On the correct entropic form for systems with power-law behaviour: the case of dissipative maps

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Abstract

Maximum entropy principle does not seem to distinguish between the use of Tsallis and Renyi entropies as either of them may be used to derive similar power-law distributions. In this paper, we address the question whether the Renyi entropy is equally suitable to describe those systems with power-law behaviour, where the use of the Tsallis entropy is relevant. We discuss a specific class of dynamical systems, namely, one dimensional dissipative maps at chaos threshold and make our study from two aspects: i) power-law sensitivity to the initial conditions and the rate of entropy increase, ii) generalized bit cumulants. We present evidence that the Tsallis entropy is the more appropriate entropic form for such studies as opposed to Renyi form.

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I. INTRODUCTION

As it is evident, an enormous variety of systems in Nature fall into the domain of validity of Boltzmann-Gibbs (BG) thermostatistics. On the other hand, it is also well-known since long that a variety of anomalous systems exists for which the powerful BG formalism exhibits serious difficulties. Some of the examples for such anomalous cases could be the long-ranged interacting systems [1], two-dimensional turbulence [2], nonmarkovian processes [3], granular matter [4], cosmology [5], high energy collisions [6], among others. In order to handle some of these anomalous systems, an attempt has been performed in 1988 [7], which is based on the generalization of the standard BG formalism by postulating a nonextensive entropy (Tsallis entropy) of the form

$$S_q \equiv \frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad (q \in \mathcal{R}) \quad , \tag{1}$$

and it recovers the standard BG entropy $S_1 = -\sum_{i=1}^{W} p_i \ln p_i$ in the $q \to 1$ limit. This generalization is usually called in the literature as Nonextensive thermostatistics or Tsallis thermostatistics (TT) [8], and it has been (still continues to be!) a matter of intensive studies during the past decade [9] both from the point of theoretical foundations of the formalism and its applications to various physical systems. The apparent success of TT gave rise to an increase of the studies with new entropy definitions [10]. At this point, we would like to mention a trivial but important issue: TT by no means intends to cover all of the anomalous physical systems where BG statistics seems to fail. It appears to be useful only for a large class of cases where power-law behaviour is observed. In this sense, alternate forms of entropy could of course be useful for other subexponential cases and therefore other possible generalizations of the standard formalism could always be projected using different entropic forms. One of these attempts, based on the Renyi entropy, has become popular nowadays [11,12]. The form of this entropy is [13]

$$S_q^R \equiv \frac{\ln \sum_i p_i^q}{1-q}. \tag{2}$$

It is an extensive quantity (unlike Tsallis entropy), concave for $0 < q < 1$ and it recovers the usual BG entropy as a special case when $q \to 1$. It is worth mentioning that the Renyi entropy has already been used in the multifractal theory [14]. On the other hand, the efforts of establishing a thermostatistics from this entropy seem to be originated from the fact that the maximum entropy principle yields the same form of distribution function (of power-law type) for both Tsallis and Renyi entropies since they are monotonic functions of each other. At this level, there is no a priori reason to choose one of these entropies as the correct entropic form for the systems which exhibit power-law type behaviour. This is not the case for the standard BG entropy, namely, everybody knows that any monotonic function of the BG entropy also gives the same distribution function of exponential type, but still the correct description of entropy is of course BG. This unclear situation of the generalized formalism seems to force some authors, for example Arimitsu and Arimitsu, to use unclear statements like "An analytical expression of probability density function of velocity fluctuation is derived with the help of the statistics based on the Tsallis entropy or the Renyi entropy" [15].

From the above discussion, a straightforward question arises naturally: Which of these two entropic forms is the correct definition for systems exhibiting power-law behaviour ? Our motivation in this paper is to try to provide an answer (at least to make the first step) to this important and intriguing question. As already seen, since maximum entropy principle cannot provide an answer, in order to accomplish this task, we shall focus on two different issues and make use of the results coming from them. These issues are i) power-law sensitivity to the initial conditions and entropy increase rates, ii) generalized bit cumulant theory.

II. POWER-LAW SENSITIVITY TO THE INITIAL CONDITIONS AND ENTROPY INCREASE RATES

As is well-known from the theory of dynamical systems (see for example [16,17]), for one-dimensional dissipative systems, it is possible to introduce a sensitivity function of type

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \ , \tag{3}$$

where $\Delta x(0)$ and $\Delta x(t)$ are discrepancies of the initial conditions at times 0 and t respectively, and it satisfies the differential equation $d\xi/dt = \lambda_1 \xi$, where λ_1 is the standard Lyapunov exponent, thus $\xi(t)$ is of exponential type ($\xi(t) = e^{\lambda_1 t}$). Consequently, when $\lambda_1 > 0$, the system is strongly sensitive to the initial conditions, whereas it is strongly insensitive if $\lambda_1 < 0$. On the other hand, there is an infinite number of points for which $\lambda_1 = 0$. This case is called the marginal case and no further analysis of this case is possible within the standard theory. It has been conjectured recently that [18] whenever λ_1 vanishes, the sensitivity function becomes of power-law type

$$\xi(t) = [1 + (1-q)\lambda_q t]^{1/(1-q)} \, , \tag{4}$$

which is the solution of the differential equation $d\xi/dt = \lambda_q \xi^q$. This new definition of the sensitivity function recovers the standard exponential one in the $q \to 1$ limit and here, λ_q is the generalized Lyapunov exponent and it scales with time inversely as λ_1 does, but this time exhibits a power-law behaviour. Consequently, if $\lambda_q > 0$ and $q < 1$ ($\lambda_q < 0$ and $q > 1$), then the system is weakly sensitive (insensitive) to the initial conditions. Most important case which is in the domain of this scenario is of course the chaos threshold. At this point (where $\lambda_1 = 0$ but $\lambda_q \neq 0$), the sensitivity function presents strong fluctuations with time and delimits the power-law growth of the upper bounds of a complex time dependence of the sensitivity to the initial conditions. These upper bounds allow us, on a log-log plot, to measure the slope ($\xi(t) \propto t^{1/(1-q)}$) from where the proper $q = q^*$ value

of the dynamical system under consideration can be estimated. This constitutes method I for determining q^* value of a dynamical system at the edge of chaos (or at any point where $\lambda_1 = 0$) and it has already been successfully used for a variety of one-dimensional dissipative maps such as the standard logistic [18], z-logistic [19], circle [20], z-circular [21], single-site [22] and asymmetric logistic [23] maps. At this stage, there is no a priori reason for relating this q index with that of the Tsallis entropy. It might well be that the corresponding entropy is the Renyi entropy since it also produces the power-law behaviour.

The above concluding statement is also valid for method II of obtaining the q^* value of a dynamical system. This method is based on the multifractal geometrical aspects of chaotic attractor at the edge of chaos. This geometry is characterized by the multifractal singularity spectrum $f(\alpha)$, which reflects the fractal dimension of the subset with singularity strenght α [14,17]. The $f(\alpha)$ function is a down-ward parabola-like concave curve and at the end points of this curve, the singularity strenght is associated with the most concentrated (α_{min}) and the most rarefied (α_{max}) regions of the attractor. The scaling behaviour of these regions has been used to propose a new scaling relation as

$$\frac{1}{1-q^*} = \frac{1}{\alpha_{min}} - \frac{1}{\alpha_{max}} \quad , \tag{5}$$

which constitutes a completely different way of calculating the proper q^* value of a given dynamical system [20]. It is evident from the previous works that, for all dissipative systems studied so far, the results of these two methods for the proper q^* value coincide within a good precision.

Finally, the connection between these q^* values and the index q of the Tsallis entropy became evident after the introduction of method III for finding q^* values . This method basically makes use of a specific generalization of Kolmogorov-Sinai (KS) entropy K_1. For a chaotic dynamical system, the rate of loss of information can be characterized by this entropy and it is defined as the increase of the BG entropy per unit time. Since the Pesin equality [24] states that $K_1 = \lambda_1$ if $\lambda_1 > 0$ and $K_1 = 0$ otherwise, it is clear that the KS entropy is deeply related to the Lyapunov exponents. Moreover, the KS entropy could be

defined through

$$K_1 \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{S_1(t)}{t}, \tag{6}$$

where t is the number of discrete time steps in case of maps, W is the number of regions in the partition of the phase space and N is the number of initial conditions that are evolving with time. Analogously, for the marginal cases, a generalized version of the KS entropy K_q has been introduced recently [18] by replacing the BG entropy with the Tsallis entropy, namely,

$$K_q \equiv \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{S_q(t)}{t}. \tag{7}$$

Consistently, the Pesin equality is expected to become $K_q = \lambda_q$ if $\lambda_q > 0$ and $K_q = 0$ otherwise. Using these ideas, the method III [25,26] is based on the conjecture that (i) a unique value of q^* exists such that K_q is finite for $q = q^*$, vanishes for $q > q^*$ and diverges for $q < q^*$; (ii) this q^* value coincides with that coming from methods I and II. Latora et al. have examined numerically the standard logistic map

$$x_{t+1} = 1 - a\, x_t^2 \ \ , \tag{8}$$

(where $0 < a \leq 2$ and $-1 \leq x_t \leq 1$) both in the chaotic region and at the chaos threshold [25]. In the chaotic region (for example, $a = 2$ case), they numerically verified that K_q vanishes (diverges) for any value of $q > 1$ ($q < 1$), being finite only for $q = 1$ (see Fig.1 of [25]), which implies that the proper q^* value is unity as expected. On the other hand, at the chaos threshold, the same structure has also been observed, but this time with an important exception: K_q vanishes (diverges) for any value of $q > q^*$ ($q < q^*$), being finite only for $q = q^* = 0.24...$ (see Fig.4 of [25]). This value coincides with the value of the proper q^* coming from other two methods. This structure has been verified not only for the standard logistic map but also a variety of other map families as well [22,23,26].

Now we are ready to proceed with our main purpose in this effort. We conjecture here that if the Tsallis and Renyi entropies are both the correct choices for a system with

power-law behaviour, then the Renyi entropy should also give us the proper q^* values whenever we use this entropic form instead of the Tsallis entropy in method III. Before demonstrating our results, let us describe the numerical procedure: Firstly, we partition the phase space into W equal cells, then we choose one of these cells and select N initial conditions all inside the chosen cell. As time evolves, these N points spread within the phase space and it gives a set $\{N_i(t)\}$ with $\sum_{i=1}^{W} N_i(t) = N$, $\forall t$, which consequently yields a set of probabilities from where one can calculate the entropy. In order to compare our results to those of Latora et al., we numerically check the Renyi entropy case for the standard logistic map both in the chaotic region and at the chaos threshold. The results are illustrated in Fig.1. Surprisingly, the expected structure has not been observed. For the chaotic region (Fig.1a), it is seen that, for $q = 1$ case, the linear entropy increase is realized but very strangely other values of q also give the same result! This shows that Renyi entropy is a much *less* sensitive function as compared to Tsallis entropy with respect to changes in the value of q. Similarly, at the chaos threshold (Fig.1b), Renyi entropy does not allow us to distinguish the correct q^* value for which a linear rate is expected. Note that rate of increase of Renyi entropy can be linear if we plot the entropy with respect to $\ln(t)$ variable, which follows due to the relation between Tsallis and Renyi entropy, (compare (1) and (2)). This is another reason for unsuitability of the Renyi entropy; for Tsallis entropy we get the result of strong chaos (BG entropy proportional to t) in the $q \to 1$ limit, but for Renyi entropy, $q \to 1$ implies that BG entropy is proportional to $\ln(t)$. These observations strongly suggest that the Renyi entropy is not the suitable entropic form for these dynamical systems which exhibit power-law behaviour at their marginal points (where $\lambda_1 = 0$). Although the results which led us to make this claim seem rather convincing, it would be no doubt convenient to seek further evidences which can strengthen this claim. This is what we shall try to do in the remainder of the paper by making use of the generalized bit cumulants.

III. GENERALIZED BIT CUMULANTS

Bit statistics is a tool to describe the complicated probability distributions, such as generated by chaotic systems [17]. In this framework, the first cumulant is the BG entropy itself, the second cumulant gives the variance (a measure of fluctuations) of the 'classical' bit number $b_i = -\ln\, p_i$, and so on. Particularly, the second bit cumulant which is a generalization of specific heat, can be applied in the context of equilibrium and non-equilibrium phase transitions [27]. The classical bit cumulants can be generalized [28] within TT formalism. They were applied to symmetric logistic and z-logistic family of maps given by

$$x_{t+1} = 1 - a|x_t|^z, \tag{9}$$

where the inflexion parameter $z > 1$, $0 < a \leq 2$ and $-1 \leq x \leq 1$ and $t = 0, 1, 2, ...$ Applications have also been found for asymmetric family of logistic maps [29], given by

$$x_{t+1} = \left\{ \begin{array}{l} 1 - a|x_t|^{z_1},\ \text{if } x_t \geq 0 \\ 1 - a|x_t|^{z_2},\ \text{if } x_t \leq 0 \end{array} \right\}, \tag{10}$$

where $z_i > 1$. We briefly summarise the motivation and also the information obtained from these applications, as we intend to apply this approach to elucidate the difference in the use of Renyi and Tsallis entropies.

The generalized second bit cumulant, which gives variance of the Tsallis bit number $-[a_i]_q \equiv \frac{(p_i)^{q-1}-1}{1-q}$, is given by

$$C_2^{(T)} = \frac{1}{(1-q)^2}\left[\sum_i (p_i)^{2q-1} - (\sum_i (p_i)^q)^2\right]. \tag{11}$$

For $q \to 1$, (11) approaches the standard bit variance,

$$C_2 = \sum_i (\ln\, p_i)^2 p_i - (\sum_i p_i \ln\, p_i)^2. \tag{12}$$

It was observed in [28] that the ratio $C_2^{(T)}/C_2$ evaluated in the chaotic region of the map (9), gives a scaling factor referred to as 'slope' in this paper (see Figs.), whose value

depends on values of q as well as the inflexion parameter z. Naturally, the slope tends to unity for $q \to 1$. Less trivially, it was observed that the slope tends to unity also for $z \to \infty$. In other words, increasing z value within the map-family, has the same effect on the quantifier 'slope' as the effect of taking to unity, the q parameter entering in its definition. Thus the otherwise free parameter q in (11) behaves analogous to the proper q^* value, which shows a monotonic decrease with increasing z values [19]. This observation was used to conjecture the behaviour of proper index q^* versus (z_1, z_2) pairs [29] in maps (10) at chaos threshold. The actual behaviour obtained from finding q^* values for these maps by using the previously introduced methods [23], agree with the conjecture of [29]. This implies that generalized second bit cumulant (11) can be consistently applied in such an analysis.

Can we have generalization of bit-cumulants based on the Renyi entropy ? If such a possibility exists, then these new quantities may be used to study the implications on the above mentioned maps, and a comparison be made of the predictions derived from Tsallis based bit-cumulants and Renyi based ones. This task of a new generalization is not a straightforward one, in view of the fact that there is no generalized bit number corresponding to the Renyi entropy, i.e., the latter cannot be written as usual mean over some bit numbers, unlike the cases of BG and Tsallis entropies. In fact, Renyi entropy can be expressed only as a kind of nonlinear average [30]. Recently, however, a step was made in this direction by one of the authors [31]. Specifically, the discrete derivative operator can be used to generate two types of bit cumulants, say type A and type B. Type A cumulants are Tsallis type, of the kind formulated in [28]. These are nonextensive quantities with respect to independent subsystems and preserve the standard relations between bit-moments and cumulants, as do the cumulants and moments of a probability distribution for a certain random variable. Type B 'cumulants' are however extensive quantites but do not preserve the standard relations between bit-moments and cumulants. Still, the approach allows us to clearly distinguish the separate origins of Tsallis and Renyi entropies, based on the use of discrete derivative. Particularly, the first

bit cumulant of type B is the Renyi entropy, the second bit cumulant is another positive valued q-generalization of the standard bit variance,

$$C_2^{(R)} = \frac{1}{(1-q)^2}\left[\ln\,\sum_i (p_i)^{2q-1} - 2\ln\,\sum_i (p_i)^q\right], \tag{13}$$

It also recovers the form (12) in the limit $q \to 1$. Thus it is interesting to see the application of (13) analogous to that of (11) for the 1-d dissipative maps. We expect that any difference in the predictions based on (13) as compared to those of (11), also reflects the separate implications for Renyi and Tsallis entropies for such systems.

In the following, we compare the results obtained from using the generalized second bit-cumulants (11) and (13). In Fig. 2a, the results for symmetric maps (9) are given. The slope which implies the ratio $C_2^{(R)}/C_2$, shows a non-monotonic behaviour with increasing z values. This can be contrasted with the monotonic decrease of the ratio $C_2^{(T)}/C_2$, as shown in Fig. 2b. Similarly, for asymmetric maps (10), we see separate trends as is clear by comparing Figs. 3a and 3b.

This makes us to conclude that while the q parameter in the definition of generalized cumulant $C_2^{(T)}$ behaves like the q^* parameter inferred from the sensitivity to initial conditions at chaos threshold (method I), the q parameter in the definition $C_2^{(R)}$ serves no such connection. Thus this generalization of the standard cumulants, which are related to Renyi entropy, are inappropriate for such studies. This clearly strengthens our claim that the Renyi entropy is not the appropriate entropic form for such dynamical systems exhibiting power-law behaviour.

IV. RATE OF ENTROPY INCREASE REVISITED

So far, we have argued and shown that although it is expected to be applicable to systems exhibiting power law behaviour, Renyi entropy is not a suitable measure to quantify the rate of entropy increase at points of power-law sensitivity in 1-d maps. However, Tsallis entropy qualifies as the appropriate measure for this purpose. A natural question

follows: Is Tsallis entropy the unique measure in this regard ? In this section, we propose that Tsallis entropy is *not* the unique measure of entropy which can give a linear rate of increase at chaos threshold. We exemplify below by proposing an alternative nonextensive entropy which can also yield a linear rate of increase. However, our conjecture is that the Tsallis entropy seems to be the unique entropy which yields the same q^* as obtained from the modified sensitivity function (4) and multifractal spectrum (5), i.e., methods I and II to determine q^*. To understand this role of the Tsallis entropy, we have to appreciate the implication of Pesin's equality.

First, let us discuss the case of strong exponential sensitivity $\lambda_1 > 0$. In (6) assuming equiprobability, $p_i = 1/M(t)$, where $M(t)$ are the number of microstates or cells occupied at time t, we have

$$K_1 = \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{1}{t} \ln M(t). \tag{14}$$

Standard Pesin's equality $K_1 = \lambda_1$ allows us to relate the above expression with exponential sensitivity function. As a result, we get

$$\xi(t) = M(t). \tag{15}$$

In other words, for strong exponential sensitivity the number of states increase linearly with the distance between trajectories for all times $t > 0$.

When $\lambda_1 = 0$, the use of modified sensitivity function (4) is suggested. Then we have $\lambda_q > 0$. Using the definition (7) of K_q and assuming equiprobability, we get

$$K_q = \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{1}{t} \frac{M(t)^{1-q} - 1}{1-q}. \tag{16}$$

Now postulating the generalized Pesin equality for the proper q^* value of the entropic index (which yields the linear rate of entropy increase), $K_{q^*} = \lambda_{q^*}$, we get from (4) and (16), the same relation as (15) even for points of power-law sensitivity. Thus the use of the Tsallis entropy in the definition of K_q ensures that the linear relation (15) is preserved. Finally, this automatically implies that the critical q^* value obtained in methods I and III are identical.

Now consider an alternate entropic form

$$\mathcal{S}_q = \frac{1 - \sum_i p_i^{(1+\ln q)}}{q-1} \quad (q \in \mathcal{R}^+) \ , \tag{17}$$

which has the same nonextensive property as the Tsallis entropy. It approximates the Tsallis entropy for values of q very near to one, and thus goes to BG entropy in the limit $q \to 1$. For this entropy, we can also obtain a linear rate of increase at chaos threshold, for certain critical value of $q = q_c$. But in this case, $q_c \neq q^*$, where q^* is calculated from methods I and II. Instead, we have $1 + \ln q_c = q^*$. A more thorough understanding of generalized Pesin's equality is required to relate such other entropies with the issue of power-law sensitivity. At the moment, we only state that there are other entropic forms which can give a linear rate of increase, but the entropic form compatible with the generalized sensitivty function (4), in the sense that methods I and III should yield same q^* values, is the Tsallis entropy.

V. CONCLUSIONS

In this study, the appropriate entropic form for systems exhibiting power-law behaviour has been tried to be determined. To accomplish this task, two different issues have been analysed. One of these issues is the sensitivity to initial conditions and the entropy increase rates, the other one is the generalized bit cumulants. From the investigation of both issues, for the dynamical systems at marginal points (where the standard Lyapunov exponent vanishes), it is verified that the suitable entropic form is the Tsallis entropy. This could be considered as the first step towards the answer of the intriguing question: What is the correct entropic form for systems exhibiting power-law behaviour ?

The reasons for the success of the Tsallis entropy, we believe, are two-fold: i) Tsallis entropy as function of time t, reproduces the correct $q \to 1$ limit of BG entropy proportional to t, ii) Tsallis entropy is a much more sensitive function than Renyi entropy with respect to changes in q value, which helps to determine the proper q^* value in method III.

A crucial property of Tsallis entropy is its nonextensivity. It is not certain whether this property plays significant role in such studies. As has been argued in [32], a conditional entropy which is nonextensive for generic values of q, becomes extensive for the proper value of $q = q^*$. It seems that, for such kind of systems with power-law behaviour, what is important is NOT to be extensive for all q values, but rather, to be extensive for only a special value of q (namely q^*), the value that gives the correct entropy for the system under investigation.

Finally, it is worth mentioning that we also show that the Tsallis entropy is not the unique entropic form which gives a linear entropy increase rate, but on the other hand, it seems to be the unique one which provides q^* values consistent with those obtained from the other two methods (i.e., from the sensitivity function and from the multifractal scaling relation).

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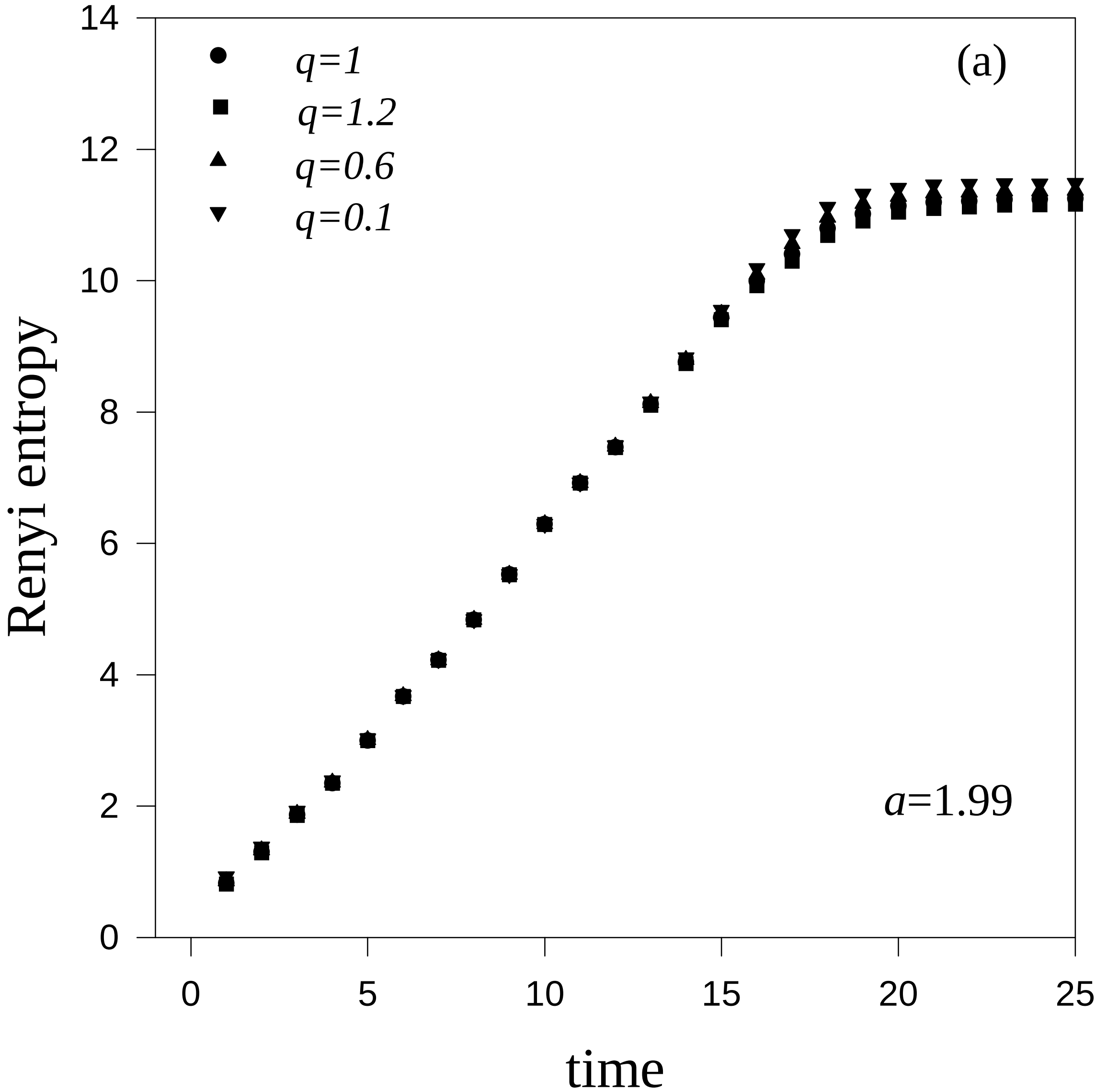
Figure Captions

Figure 1 - Time evolution of the Renyi entropy at the chaotic region (a) and at the chaos threshold (b) for various q values. Inset of (b): The behaviour of the Renyi entropy for $\ln t$.

Figure 2 - The behaviour of the slope as a function of the map inflexion parameter z for the Renyi (a) and Tsallis (b) cases.

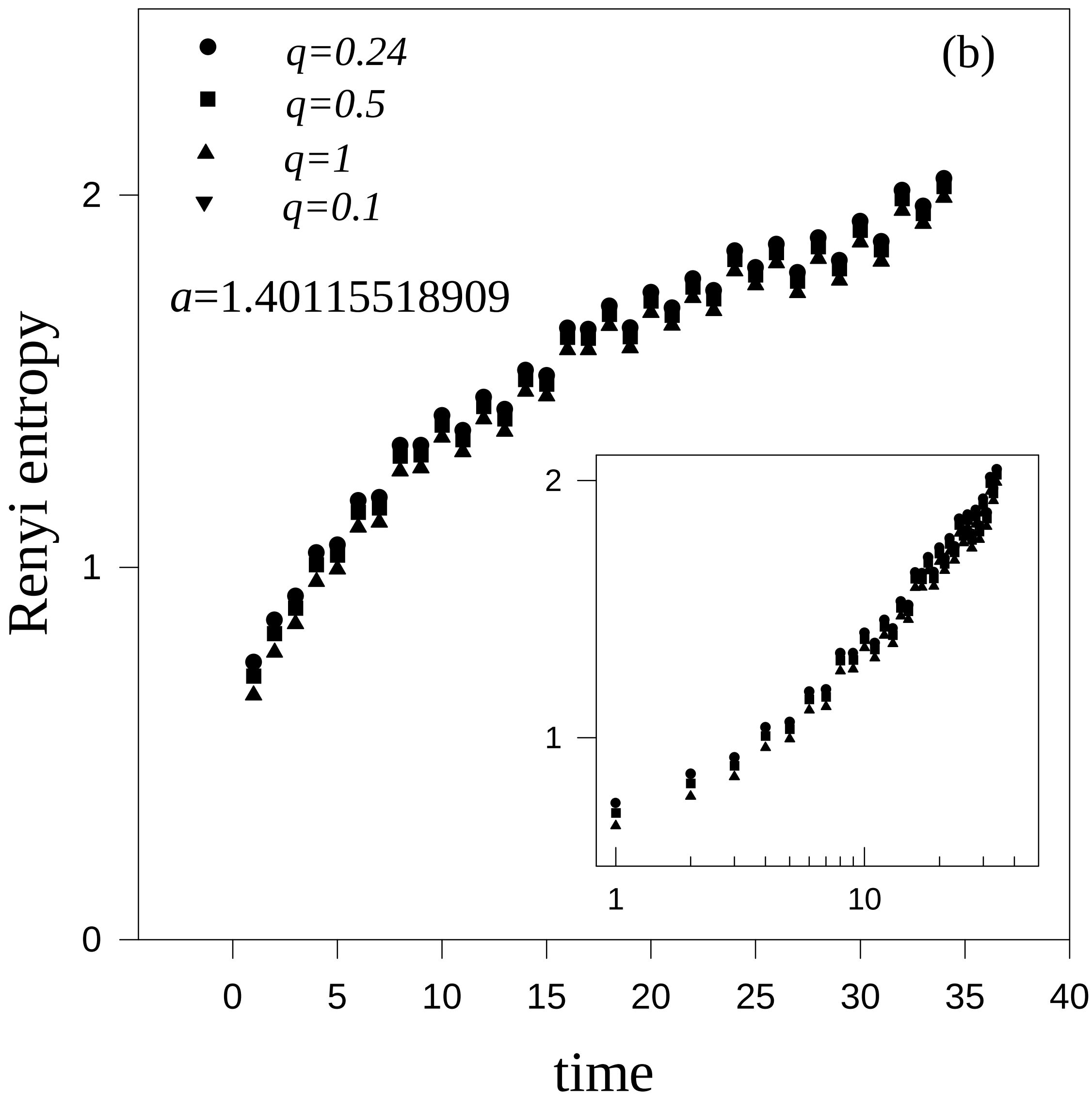
Figure 3 - The behaviour of the slope as a function of the map inflexion parameter pairs (z_1, z_2) for the Renyi (a) and Tsallis (b) cases.

Fig 1a



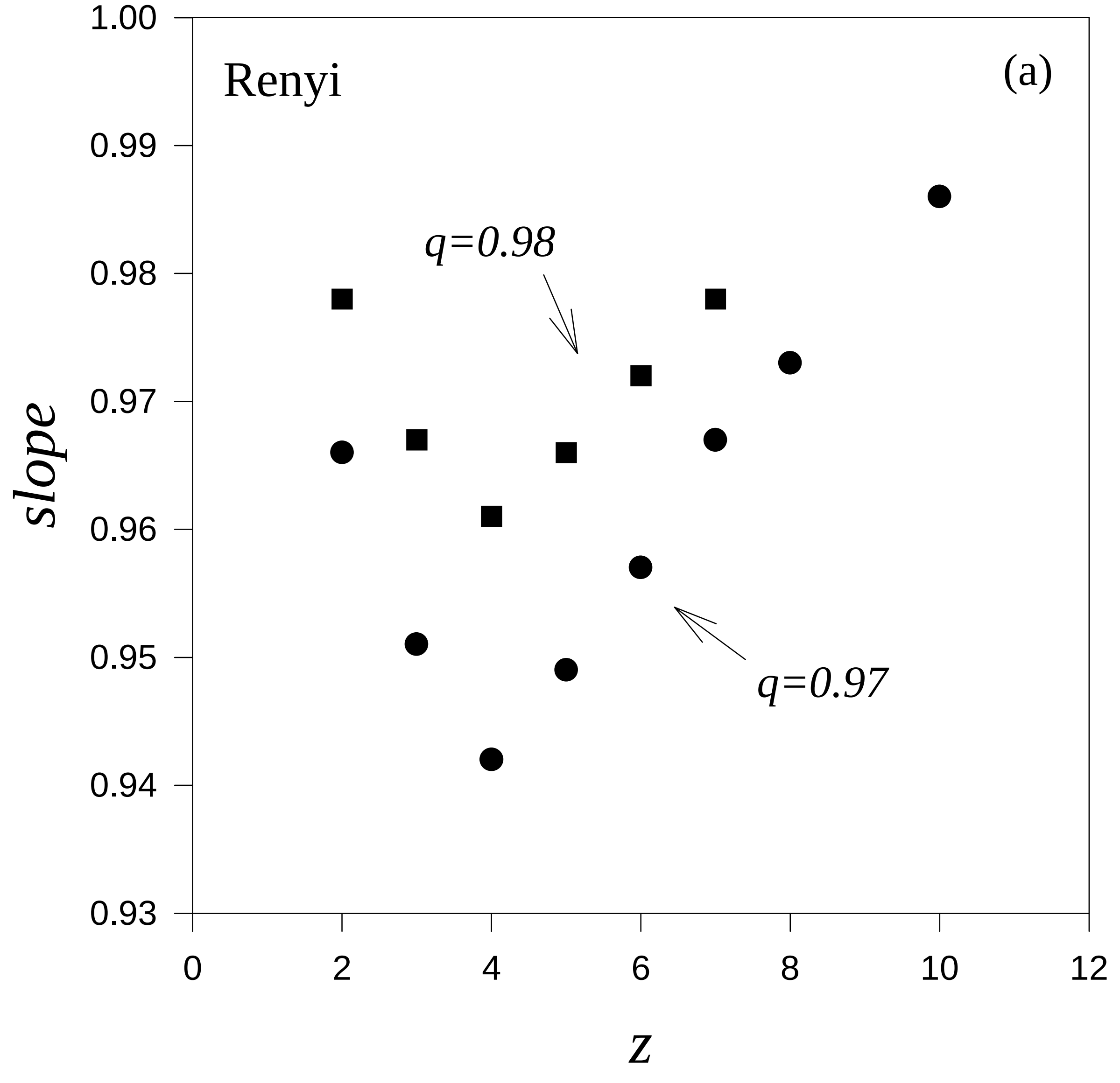
q=1
q=1.2
q=0.6
q=0.1
(a)
a=1.99
Renyi entropy
time
0
2
4
6
8
10
12
14
0
5
10
15
20
25

Fig 1b



q=0.24
q=0.5
q=1
q=0.1
(b)
a=1.40115518909
Renyi entropy
time
0
1
2
0
5
10
15
20
25
30
35
40
1
2
1
10

Fig 2a



1.00
0.99
0.98
0.97
0.96
0.95
0.94
0.93
slope
Renyi
(a)
q=0.98
q=0.97
0
2
4
6
8
10
12
z

Fig 2b

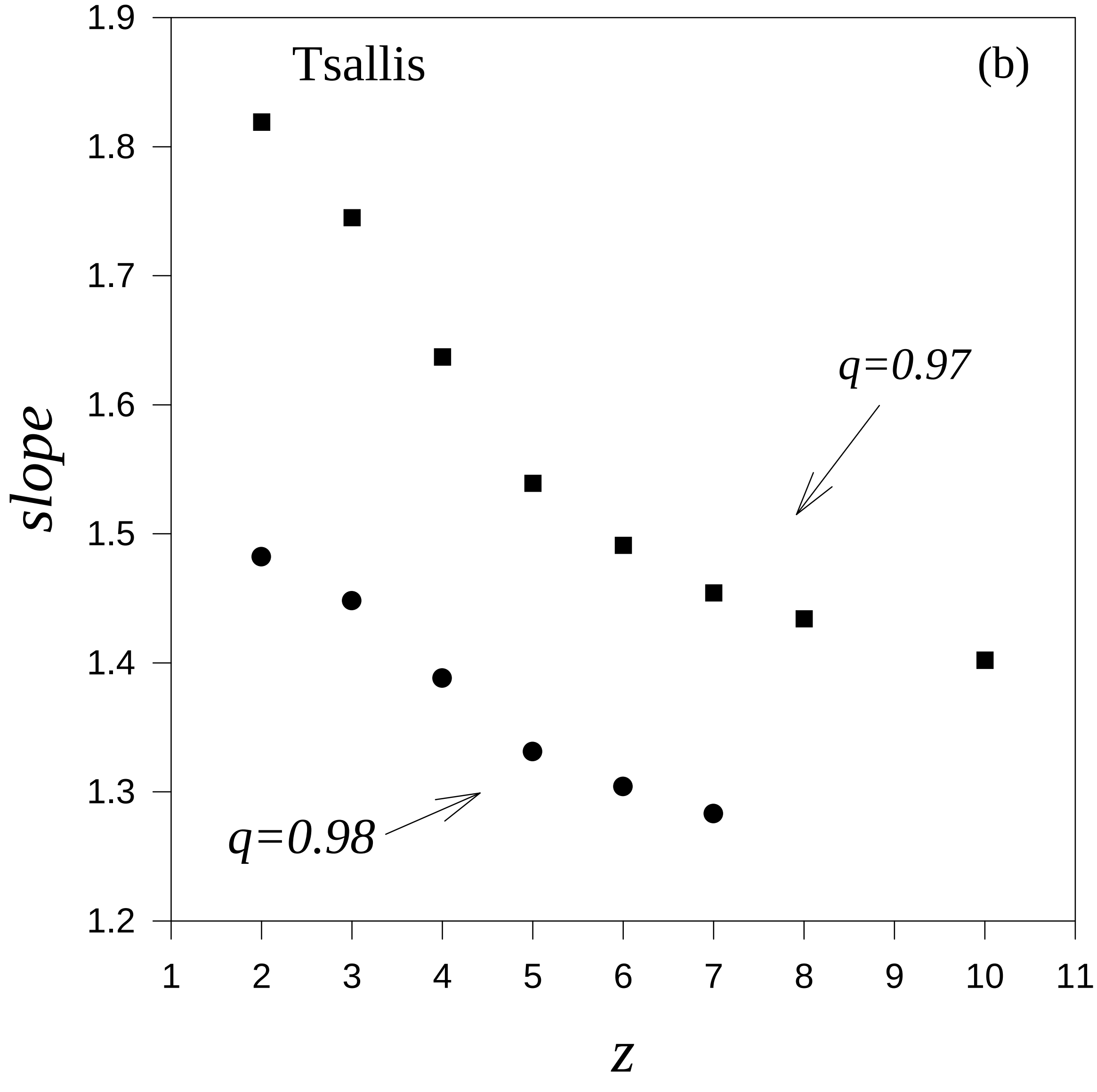

Tsallis
(b)
q=0.97
q=0.98
slope
z
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
1
2
3
4
5
6
7
8
9
10
11

Fig 3a

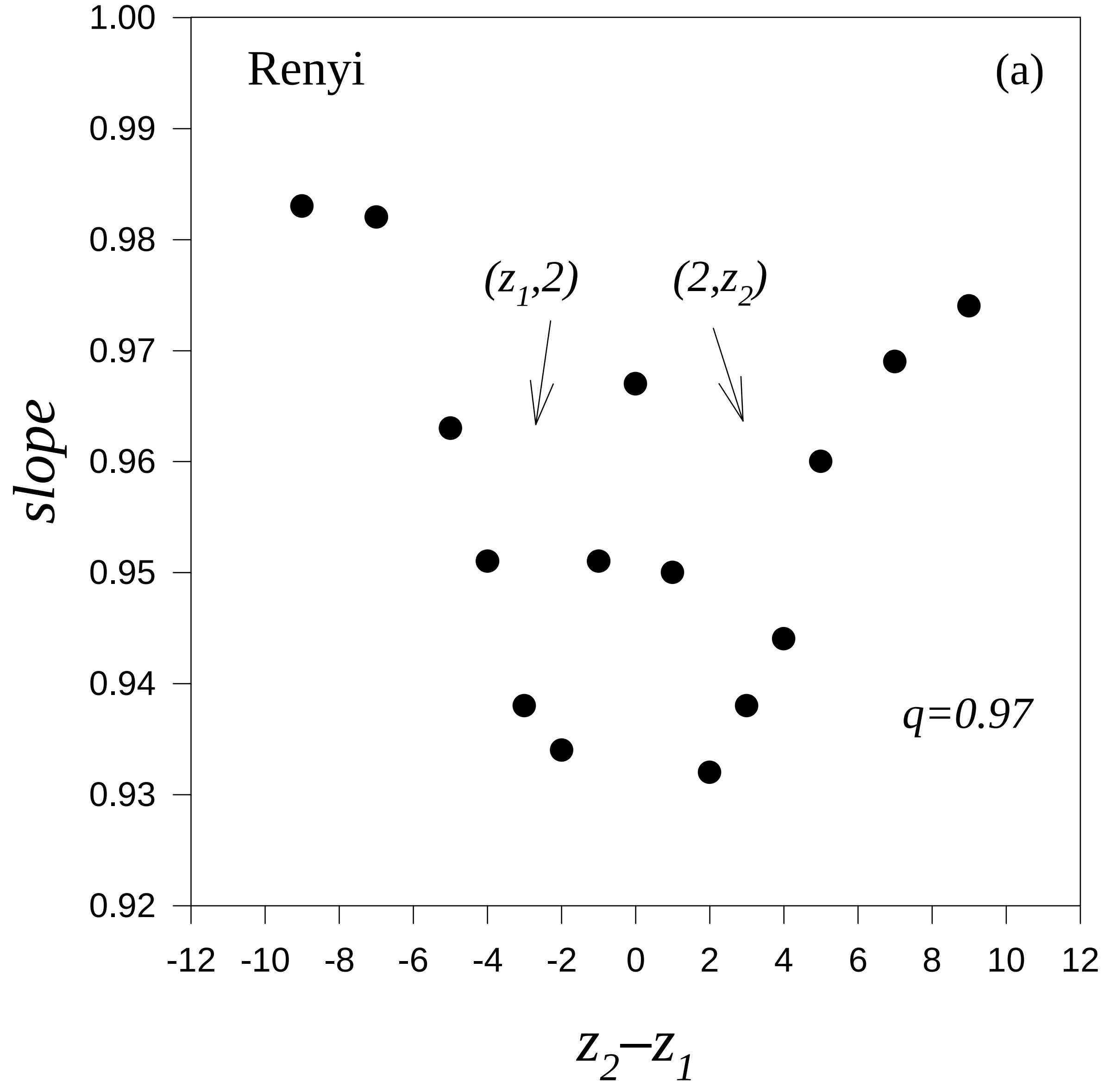
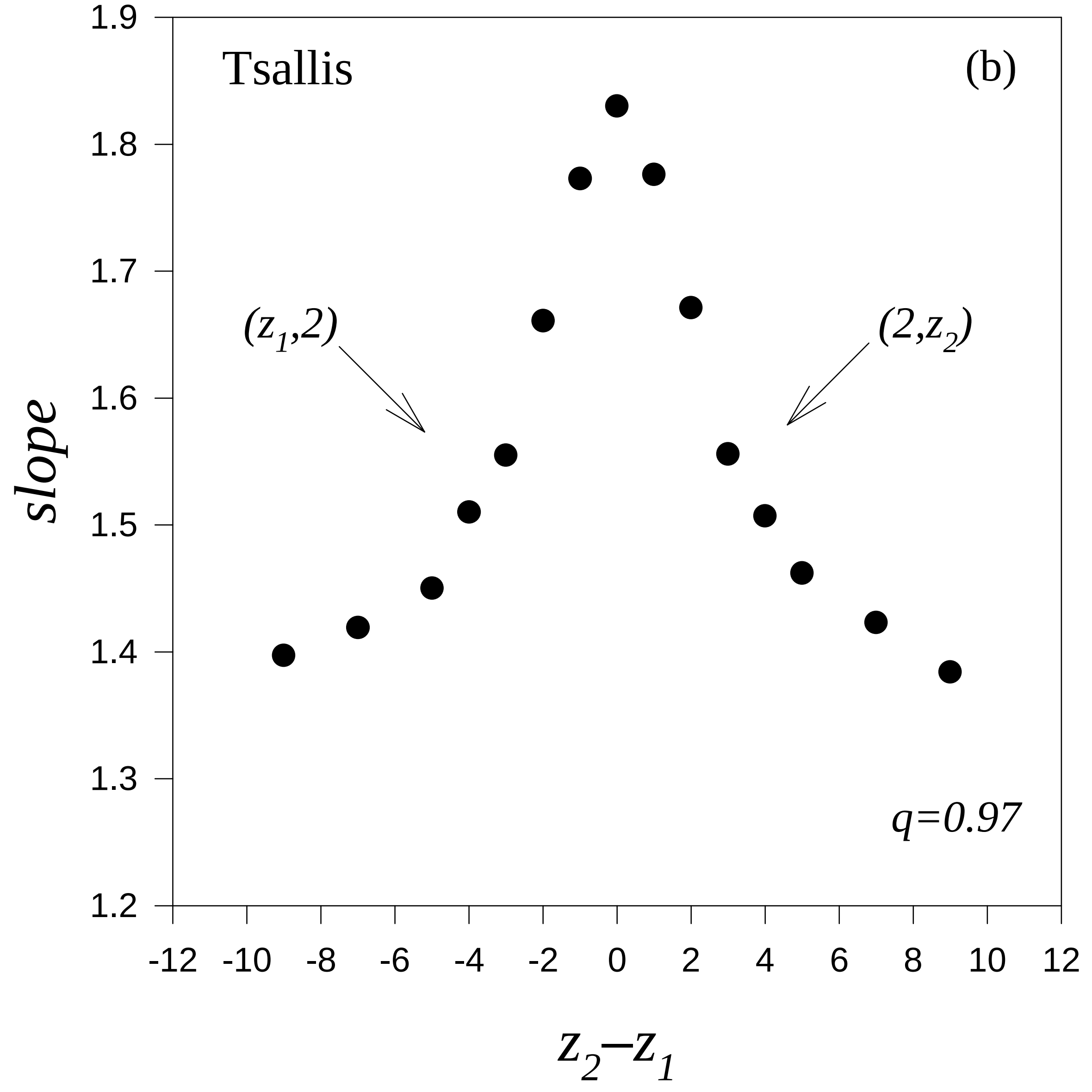

Renyi
(a)
$(z_1,2)$
$(2,z_2)$
$q=0.97$
slope
1.00
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
-12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12
$z_2 - z_1$

Fig 3b



Tsallis
(b)
$(z_1,2)$
$(2,z_2)$
q=0.97
slope
z_2-z_1
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
-12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12